Exam. Code : 211004
Subject Code : 4635

M.Sc. (Mathematics) 4th Semester

TOPOLOGY—II

Paper—MATH-582

Time Allowed—2 Hours] [Maximum Marks—100

Note :— Attempt any **four** questions. All questions carry equal marks.

1. (a) Let $\{x_\alpha\}_{\alpha\in J}$ be a family of topological spaces. If the product space $\prod_{\alpha\in J} X_\alpha$ is normal, show that each X_α is normal. Check whether $\mathbb{R}^\omega$ is normal in the product topology.

   (b) Prove that a linear Continuum is normal.

2. (a) Show that the acountable product of $\mathbb{R}$ with itself is completely regular in the box topology.

   (b) Let X be a completely regular space, let A and B be disjoint closed subsets of X. Show that if A is compact, then there is a continuous function $f: X\rightarrow[0, 1]$ such that

   $$f(A) = \{0\} \text{ and } f(B) = \{1\}.$$

3. (a) Let $\mathcal{A}$ be an open covering of a metric space (X, d). If X is compact, prove that there exists a $\delta > 0$ such that for each subset B of X, having diameter less than δ, there exists an element of $\mathcal{A}$ containing B.

(b) Characterize all compact subspaces of $\mathbb{R}^n$ in the product topology.

4. (a) Let X be a Hausdörff space. Show that X is locally compact if and only if given $x \in X$, and given a neighborhood U of x, there is a neighbourhood V of x such $\overline{V}$ is compact and $\overline{V} \subseteq U$.

(b) Let X be a set; let $\mathcal{A}$ be a collection of subsets of X having FIP (Finite Intersection Property) :

(i) Show that there exists a collection $\mathcal{D}$ of subsets of X having FIP such that $\mathcal{D} \supseteq \mathcal{A}$ and $\mathcal{D}$ is maximal such collection of subsets of X.

(ii) Let $\mathcal{D}$ be the maximal collection of subsets of X having FIP. Show that $\mathcal{D}$ is closed undertaking finite intersections. Further if $A \subseteq X$ such that $A \cap D \neq \phi$ for every $D \in \mathcal{D}$, show that $A \in \mathcal{D}$.

5. (a) Let Y be a compactification of X and $\beta(x)$ denotes the stone–ťech compactification of X. Show that there exists a continuous surjective map $g : \beta(X) \to Y$ s.t. g is closed and g equals identity on X.

(b) State and prove the Urysohn Metrication Theorem.

6. (a) Prove that a topological space X is completely regular if and only if it is homeomorphic to a subspace of $[0, 1]^J$ forsome subset J of $\mathbb{R}$.

(b) Let X be a topological space in which one point sets are closed. Let $\{f_\alpha\}_{\alpha\in J}$ is an indexed family of continuous functions $f_\alpha : X \to \mathbb{R}$ such that for each point x_0 of X and each open set $U \ni x_0$, there is an index α s.t. $f_\alpha(x_0) > 0$ and $f_\alpha(X - U) = \{0\}$. Prove that the map $F : X \to \mathbb{R}^J$ defined by $F(x) = (f_\alpha(x))_{\alpha\in J}$ is an imbedding of X into $\mathbb{R}^J$.

7. (a) Let A be a subset of a topological space. Let $x \in X$. Prove that $x \in \overline{A}$ if and only if there is a "net" of points of A converging to x.

(b) Let $f : X \to Y$. Show that f is continuum if and only if for every convergent net (x_α) in X converging to $x \in X$, implies that the net $(f(x_\alpha))_{\alpha\in J}$ converges to f(x).

8. (a) Write short notes on :—

(i) Filters

(ii) Ultra filters and compactness.

(b) Prove that a topological space X is compact if and only if every "net" in X has a convergent "subnet".